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Two Approaches to Coupling Classical and Quantum Variables

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We address the issue of coupling variables which are essentially classical to variables that are quantum. Two approaches are discussed. In the first, continuous quantum measurement theory is used to construct a phenomenological description of the interaction of a quasiclassical variable X with a quantum variable x, where the quasiclassical nature of X is assumed to have come about as a result of decoherence. The state of the quantum subsystem evolves according to the stochastic nonlinear Schrödinger equation of a continuously measured system, and the classical system couples to a stochastic c-number $\bar{x}(t)$ representing the imprecisely measured value of x. The theory gives intuitively sensible results even when the quantum system starts out in a superposition of well-separated localized states. The second approach involves a derivation of an effective theory from the underlying quantum theory of the combined quasiclassical–quantum system, and uses the decoherent histories approach to quantum theory.

1. INTRODUCTION

What happens when a classical system interacts with a quantum system in a nontrivial superposition state? Quantum field theory in curved spacetime is an example of a number of situations where one would like to know the answer to this question. There, the effect of the quantized matter field on the classical gravitational field is often assessed using the semiclassical Einstein equations [1, 2]

$$G_{\mu\nu} = 8\pi G\langle T_{\mu\nu}\rangle \tag{1.1}$$

The left-hand side is the Einstein tensor of the classical metric field $g_{\mu\nu}$ and the right-hand side is the expectation value of the energy-momentum tensor of a quantum field.

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Although we do not yet have the complete, workable theory of quantum gravity required to derive an equation like (1.1), on general grounds it is clear that it is unlikely to be valid unless the fluctuations in $T_{\mu\nu}$ are small [3–5]. Indeed, (1.1) fails to give intuitively sensible results when the matter field is in a superposition of localized states [6, 7]. In particular, when the quantum state of the matter field consists of a superposition of two well-separated localized states, Eq. (1.1) suggests that the gravitational field couples to the average energy density of the two states, while physical intuition suggests that the gravitational field feels the energy of one or other of the localized matter states, with some probability. It is by no means obvious, however, that we have to resort to quantum gravity to accommodate such nontrivial matter states. This leads one to ask whether there exists a semiclassical theory with a much wider range of validity than (1.1) which gives intuitively reasonable results for nontrivial superposition states for the matter field.

The aim of this contribution is to describe two related approaches to coupling classical and quantum variables which go far beyond the naive mean-field equations, and produce intuitively sensible results in the key case of superposition states. The full problem of the semiclassical Einstein equations (1.1) will not be addressed. Rather, we will concentrate on a simple model in which the scheme is easily presented and perhaps verified. Of course, many previous authors have tackled this problem [8–11]. What is perhaps new in the present approach compared to previous ones is the explicit incorporation of the notion of decoherence to ensure that the "classical" system really is classical. (See, however, ref. 8, for some earlier comments along these lines.)

Our considerations will be based entirely on the following simple model, consisting of a classical particle with position X in a potential $V(X)$ coupled to a harmonic oscillator with position x which will later be quantized. The action is

$$S = \int dt \left(\frac{1}{2} M\dot{X}^2 - V(X) + \frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2 - \lambda X x \right) \tag{1.2}$$

Hence the classical equations of motion are

$$M\ddot{X} + V'(X) + \lambda x = 0 \tag{1.3}$$

$$m\ddot{x} + m\omega^2 x + \lambda X = 0 \tag{1.4}$$

The naive mean-field approach involves replacing (1.3) with the equation

$$M\ddot{X} + V'(X) + \lambda\langle\psi|\hat{x}|\psi\rangle = 0 \tag{1.5}$$

and replacing (1.4) with the Schrödinger equation

$$\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar}(\hat{H}_0 + \lambda X\hat{x})|\psi\rangle \tag{1.6}$$

for the quantum particle. $\hat{H}_0$ is the Hamiltonian of the quantum particle (in this case a harmonic oscillator) and $-X(t)$ is regarded as an external classical force. As stated above, the scheme (1.5), (1.6) is unlikely to have a very wide range of validity.

Generally, for a quantum system with wave function $\psi(x)$, there will be a nonzero probability for x to take any one of a range of values, and the expectation value $\langle\hat{x}\rangle$ [as in Eq. (1.5)] will not be representative of the distribution of x (unless the distribution just happens to be peaked about its expectation value). One would therefore expect the classical system to be stochastically influenced by the quantum system and follow one of an ensemble trajectories. To be precise, we expect an improved version of (1.5) to be of the form

$$M\ddot{X} + V'(X) + \lambda\bar{x}(t) = 0 \tag{1.7}$$

where $\bar{x}(t)$ is now a classical *stochastic* variable, whose probability distribution is determined by the dynamics and quantum state of the quantum particle.

The purpose of this paper is to describe two different but related approaches to coupling classical and quantum variables, both of which lead to an equation of the form (1.7) and both of which yield an explicit probability distribution for $\bar{x}(t)$. The first approach (which was developed in collaboration with Lajos Diósi) is a phenomenological scheme based on continuous quantum measurement theory. The second is a more fundamentally based scheme, derived using the decoherent histories approach to quantum theory. This work is based on two published papers [12, 13].

Taking the second of these schemes first, the question of coupling classical variables to quantum variables is intimately connected to the question of how certain variables become classical in the first place. We adopt the point of view that there are no *fundamentally* classical systems in the world, only quantum systems that are effectively classical under certain conditions. The most comprehensive approach to obtaining generalizations of the semiclassical scheme (1.5), (1.6) therefore consists in starting from the underlying quantum theory of the whole composite system and then *deriving* the effective form of that theory under the conditions in which one of the subsystems is effectively classical. The most important condition that needs to be satisfied for a subsystem to be effectively classical is *decoherence* — interference between histories of certain types of variables (in this case position) must be destroyed (see, for example refs. 14 and 15). Decoherence is typically brought about by some kind of coarse-graining procedure, of which perhaps the most commonly used procedure is to couple to a large environment (typically a heat bath) and then trace it out. The resulting decoherent variables are often referred to as quasiclassical, a nomenclature we shall adopt. Quasiclassical variables follow classical trajectories, but modified by fluctuations induced by the environment that decoherered them. For sufficiently massive particles, these fluctuations have negligible effect.

A derivation of an effective theory of coupled quasiclassical and quantum variables therefore involves a three-component quantum system consisting of a ("to be quasiclassical") particle with position X, coupled to an environment which is traced out to render X quasiclassical, and also coupled to the position x of another ("quantum") particle (not necessarily coupled to the environment). In Section 3, we will show, in the context of a particular model, how such an effective theory may be derived using the decoherent histories approach to quantum theory.

Emergent classicality is, however, a widespread and generic phenomenon. It has been demonstrated in a wide variety of different models using a variety of different approaches to decoherence. This suggests that it ought to be possible to write down directly a phenomenological model describing the coupling of the quasiclassical variable X to the quantum variable x, but without having to appeal to the full details of a specific decoherence calculation. Differently put, it is of interest to determine the minimal elaboration required of Eqs. (1.5), (1.6) to obtain a viable scheme of coupled classical–quantum variables. Such a scheme would also have the advantage that it may be valid when the underlying quantum theory is not particularly manageable or even not known (as may be the case for gravity).

For these reasons, in Section 2, a more phenomenological approach to classical quantum couplings is presented. This approach is based on the observation that there already exists a partial description of classical–quantum couplings in the form of continuous quantum measurement theory. This existing structure, together with a heurisitic appreciation of decoherence, leads to the desired phenomenological scheme. The idea is to think of the quasiclassical particle as in some sense "measuring" the quantum particle's position and responding to the measured c-number result $\bar{x}$ (a precursor to this idea may be found in ref. 16). In this approach, the decoherence of the quasiclassical particle is not modeled explicitly, but an appeal is made to general known features of the decoherence process where necessary. In particular, the assumed decoherence ensures that the quasiclassical particle remains quasiclassical (although may be stochastically influenced) even when it interacts with the quantum particle in a nontrivial superposition.

The two models are summarized in Section 4.

2. CLASSICAL–QUANTUM COUPLINGS VIA CONTINUOUS QUANTUM MEASUREMENT THEORY

As stated in the Introduction, the first approach to coupling classical and quantum variables is a phenomenological scheme using continuous quantum measurement theory. The basic idea is to think of the classical variable as in some sense "measuring" the quantum particle and responding to the measured c-number result.

Consider, therefore, the consequences of standard quantum measurement theory for the evolution of the coupled quasiclassical and quantum systems over a small interval of time δt. The state $|\psi\rangle$ of the quantum system will evolve, as a result of the measurement, into the (unnormalized) state

$$|\Psi_{\bar{x}}\rangle = \hat{P}_{\bar{x}}\, e^{-i\hat{H}\delta t}|\psi\rangle \tag{2.1}$$

where $\hat{H} = \hat{H}_0 + \lambda X\hat{x}$ and $\hat{P}_{\bar{x}}$ is a projection operator which asks whether the position of the quantum particle is $\bar{x}$, to within some precision. [If the classical system couples to some operator of the quantum system other than position, e.g., momentum, then the projection operator in (2.1) is changed accordingly, e.g., to a momentum projector.] The probability that the measurement yields the result $\bar{x}$ is given by $\langle\Psi_{\bar{x}}|\Psi_{\bar{x}}\rangle$. It is then natural to suppose that the classical particle, in responding to the measured result, will evolve during this small time interval according to the equation of motion

$$M\ddot{X} + V'(X) + \lambda\bar{x} = 0 \tag{2.2}$$

with probability $\langle\Psi_{\bar{x}}|\Psi_{\bar{x}}\rangle$.

Now we would like to repeat the process for an arbitrary number of time steps and then take the continuum limit. If $\hat{P}_{\bar{x}}$ is an exact projection operator, i.e., one for which $\hat{P}_{\bar{x}}^2 = \hat{P}_{\bar{x}}$, the continuum limit is trivial and of no interest (this is the watchdog effect). However, standard quantum measurement theory has been generalized to a well-defined and nontrivial process that acts continuously in time by replacing $\hat{P}_{\bar{x}}$ with a positive-operator-valued measure (POVM) [16–19]. The simplest example, which we use here, is a Gaussian,

$$\hat{P}_{\bar{x}} = \frac{1}{(2\pi\Delta^2)^{1/2}} \exp\left(-\frac{(\hat{x} - \bar{x})^2}{2\Delta^2}\right) \tag{2.3}$$

and the continuum limit involves taking $\Delta \to \infty$ as $\delta t \to 0$ in such a way that $\Delta^2\delta t$ is held constant. The evolution of the wave function of the quantum system is then conveniently expressed in terms of a path-integral expression for the unnormalized wave function:

$$\Psi_{[\bar{x}(t)]}(x', t') = \int \mathcal{D}x \exp\left(\frac{i}{\hbar}\int_0^{t'} dt \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2 - \lambda x X\right)\right)$$
$$\times \exp\left(-\int_0^{t'} dt \frac{(x - \bar{x})^2}{4\sigma^2}\right)\Psi(x_0, 0) \tag{2.4}$$

Here, the integral is over paths $x(t)$ satisfying $x(0) = x_0$ and $x(t') = x'$. The classical particle at each moment of time evolves according to Eq. (2.2), where the functional probability distribution of the entire measured path $\bar{x}(t)$ takes the form

$$p[\bar{x}(t)] = \langle\Psi_{[\bar{x}(t)]}|\Psi_{[\bar{x}(t)]}\rangle \tag{2.5}$$

[The parameter σ in Eq. (2.5), representing the width of the effective "measurement" of the particle by the classical system, will be discussed below.]

The scheme is therefore as follows. We solve the equations (2.2) and (2.4) where $\bar{x}(t)$ is regarded as a stochastic variable whose probability distribution is given by (2.5). The final result is an ensemble of $\bar{x}$-dependent classical and quantum trajectories respectively for the two particles, with an interdependent probability distribution.

It turns out that this system (2.2), (2.4), (2.5) can be rewritten in such a way that brings it closer to the form of the naive mean-field equations (1.5), (1.6). The basic issue is that Eq. (2.5) gives the probability for an entire history of measured alternatives, $\bar{x}(t)$. Yet the naive mean-field equations (1.5), (1.6) are evolution equations defined at each moment of time. Fortunately, the system (2.2), (2.4), (2.5) may be rewritten as follows. Consider the basic process (2.1) with the Gaussian projector (2.3), but in addition let the state vector be normalized at each time step. Then denoting the normalized state at each time by $|\psi\rangle$, and taking the continuum limit in the manner indicated above, it is readily shown [19] that $|\psi\rangle$ obeys a stochastic nonlinear equation describing a system undergoing continuous measurement:

$$\frac{d}{dt}|\psi\rangle = \left(-\frac{i}{\hbar}(\hat{H}_0 + \lambda X\hat{x}) - \frac{1}{4\sigma^2}(\hat{x} - \langle\hat{x}\rangle)^2\right)|\psi\rangle$$
$$+ \frac{1}{2\sigma}(\hat{x} - \langle\hat{x}\rangle)|\psi\rangle\eta(t) \tag{2.6}$$

Here, $\eta(t)$ is the standard Gaussian white noise, with linear and quadratic means,

$$M[\eta(t)] = 0, \qquad M[\eta(t)\eta(t')] = \delta(t - t') \tag{2.7}$$

where $M(\cdots)$ denotes stochastic averaging. The noise terms are to be interpreted in the sense of Ito. The measured value $\bar{x}$ is then related to η by

$$\bar{x} = \langle\psi|\hat{x}|\psi\rangle + \sigma\eta(t) \tag{2.8}$$

Hence the final form of Eq. (2.2) [replacing Eq. (1.5)] is

$$M\ddot{X} + V'(X) + \lambda\langle\psi|\hat{x}|\psi\rangle + \lambda\sigma\eta(t) = 0 \tag{2.9}$$

and (1.6) is replaced by the stochastic nonlinear equation (2.6).

Turn now to the question of the value of the parameter σ. As discussed above, the quasiclassical particle suffers fluctuations as a result of interacting with the environment that decohered it. This must still be true even when it is not coupled to the quantum particle. We can therefore fix σ by demanding that in Eq. (2.9), the term $\lambda\sigma\eta(t)$, in the limit $\lambda \to 0$, describe the environmentally induced fluctuations suffered by the classical particle. This forces us to choose σ to be proportional to λ^{-1}. Further information on the form of σ requires more specific details about the environment. In the particular but frequently studied case of a thermal environment, the random force should be $\sqrt{2M\gamma k_B T}\eta(t)$, in order to coincide with the standard Langevin equation of classical Brownian motion. From this we deduce that $\sigma^2 = 2M\gamma k_B T/\lambda^2$. The result is not hard to understand. Because of the environmentally induced fluctuations it suffers, the quasiclassical particle is necessarily limited in the precision with which it can "measure" the quantum particle, hence the width σ of the "measurement" is related to the fluctuations of the quasiclassical particle.

The formal solution to (2.6) describes a family of pure states, $|\psi\rangle = |\psi_{[\eta(t)]}\rangle$, one for each choice of function $\eta(t)$. Correspondingly in Eq. (2.9), with $|\psi\rangle = |\psi_{[\eta(t)]}\rangle$ inserted in the pure state expectation value, there is one evolution equation for each $\eta(t)$. For fixed initial data $|\psi_0\rangle$, X_0, and $\dot{X}_0$, Eqs. (2.6) and (2.9) therefore describe an ensemble of quantum and classical trajectories $(|\psi_{[\eta(t)]}\rangle, X_{[\eta(t)]}(t))$, with members labeled by $\eta(t)$. The probability for each member of the ensemble is that implied by the probability distribution of $\eta(t)$ [implicit in Eq. (2.7)].

There are two differences between the system (2.6)–(2.9) and the naive mean-field equations (1.5), (1.6). One is the noise term η. In Eq. (2.9) [as compared to Eq. (1.5)] the noise clearly describes an additional (completely uncorrelated) random force. This sort of modification to the semiclassical Einstein equations has been considered previously [5, 20].

More important is the novelty that the state $|\psi\rangle$ evolves according to the stochastic nonlinear equation (2.6), and hence its evolution is very different to that under the usual Schrödinger equation, (1.6). In particular, it may be shown that all solutions to (2.6) undergo *localization* [21–24] on a time scale which might be extremely short compared to the oscillator's freqency ω. That is, every initial state rapidly evolves to a generalized coherent state centered around values $\langle\hat{x}\rangle$, $\langle\hat{p}\rangle$ undergoing classical Brownian motion. [The results cited above are readily extended to the case here in which the Hamiltonian contains a linear coupling to an external force $-X(t)$.] Which particular solution the state becomes centered around depends statistically on the initial state of the system. For an initial state consisting of a superposition of well-separated coherent states,

$$|\psi\rangle = \alpha_1|x_1 p_1\rangle + \alpha_2|x_2 p_2\rangle \tag{2.10}$$

the state after localization time will, with probability $|\alpha_1|^2$, be as if the initial state were just $|x_1 p_1\rangle$, and with probability $|\alpha_2|^2$, will be as if the initial state were just $|x_2 p_2\rangle$ [24]. The localization time $\sim 1/\sigma^2(x_1 - x_2)^2$ becomes, with our previous choice $\sigma^2 \sim M\gamma k_B T/\lambda^2$, very short indeed if the classical particle has a large mass M.

Hence in the new semiclassical equations (2.6)–(2.9), effectively what happens is that we solve separately for the two initial states $|x_1 p_1\rangle$ and $|x_2 p_2\rangle$, and the classical particle then follows the first solution with probability $|\alpha_1|^2$ and the second with probability $|\alpha_2|^2$. In simple terms, therefore, an almost classical system interacting through position with a quantum system in a superposition state (2.10) "sees" one or other of the superposition states with some probability, and not the mean position of the entire state. This is the key case for which the naive mean-field equations fail to give intuitively sensible results [6, 25], and this is the main result of the model.

It is interesting to note that nonlinear Schrödinger equations have been considered before in the context of the semiclassical Einstein equations [7, 26] because the combined system consiting of (1.1) together with the Schrödinger equation for the quantum state is nonlinear. The motivation here is rather different. The equation (2.6) used here arises because it gives a phenomenological description of continuous measurement.

Note that our classical stochastic equations (2.9) do not involve dissipation, as one might expect. Dissipation will arise in Eq. (2.9) if the model of the measurement process, Eqs. (2.1), (2.3), is extended to include feedback forces (see, for example, ref. 18). This would modify our scheme, but does not alter it in a fundamental way. Here, for brevity, we have worked in the commonly used and instructive approximation of negligible dissipation.

3. DERIVATION OF AN EFFECTIVE THEORY FROM DECOHERENT HISTORIES

In this section we discuss a more specific but more fundamental theory of classical–quantum couplings, which is *derived* from the underlying quantum theory of the whole composite system. As argued in the Introduction, classicality of a particle arises as a result of decoherence due to the interaction with an environment. We therefore consider a three-component composite system consisting of a particle with coordinate X (eventually to be the quasiclassical system) coupled to an environment consisting of an infinite number of harmonic oscillators with coordinates q_n in a thermal state. The classical particle is also coupled to a small quantum particle with coordinate x. The total action is

$$\begin{aligned} S = & \int dt \left(\frac{1}{2} M\dot{X}^2 - V(X) \right) \\ & + \int dt \sum_n \left(\frac{1}{2} m_n \dot{q}_n^2 - \frac{1}{2} m_n \omega_n^2 q_n^2 + c_n q_n X \right) \\ & + \int dt \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2 - \lambda X x \right) \end{aligned} \tag{3.1}$$

We will analyze this system using the decoherent histories approach to quantum theory. This approach is reviewed in detail elsewhere, so here we summarize only the essential parts of it that will be needed for this calculation [14, 15, 27–32].

It is not difficult to see why the decoherent histories approach is useful in this context [33]. We would like to derive an effective evolution equation for the variable X, which we expect to be approximately classical motion, plus a stochastic influence from the quantum system to which it couples. We can see whether a particle follows such a trajectory by computing the probability for a history of positions distributed in time, i.e., an object of the form $p(X_1, t_1, X_2, t_2, X_3, t_3, \ldots)$. This is the probability that the particle is at the approximate position X_1 at t_1, at X_2 at t_2, and so on. Because of quantum interference, probabilities cannot immediately be assigned to histories. We therefore need a mechanism to produce decoherence of the particle, hence the coupling to the environment.

To compute the probability for a history of particle positions, we may take as a starting point Feynman's assertion that the amplitude for a history $X(t)$ is proportional to $\exp\{(i/\hbar)S[X(t)]\}$, where $S[X(t)]$ is the action for the path [34]. The amplitude for a restricted type of path (such as one close to a classical trajectory) is obtained by summing over all paths satisfying the restrictions. So, for example, the amplitude to start at X_0, pass through gates labeled by α_1, α_2 at times t_1, t_2, and end up at x_f is given by

$$\mathcal{A}(X_0, \alpha_1, \alpha_2, X_f) = \int_{\alpha_1 \alpha_2} \mathcal{D}X(t) \exp\left(\frac{i}{\hbar} S[X(t)] \right) \tag{3.2}$$

where the sum is over all paths satisfying the stated restrictions. The candidate expression for the probability is then obtained by attaching an initial state, squaring, and summing over final values of X_f:

$$p(\alpha_1, \alpha_2) = \int dX_f \left| \int dX_0 \, \mathcal{A}(X_0, \alpha_1, \alpha_2, X_f) \Psi_0(X_0) \right|^2 \tag{3.3}$$

This formula, and indeed the probability for any set of histories characterized by restricted paths in configuration space, may be rewritten quite generally as

$$p(\alpha) = \int_\alpha \mathcal{D}X(t) \int_\alpha \mathcal{D}Y(t) \exp\left(\frac{i}{\hbar} \mathrm{S[X]} - \frac{i}{\hbar} \mathrm{S[Y]}\right) \rho_0(X_0, Y_0) \tag{3.4}$$

where ρ_0 is the initial state. Here α denotes the restrictions on the paths.

Probabilities for histories defined in this way are nonnegative and properly normalized. But an important condition that they must satisfy is additivity on disjoint regions of sample space. That is, if α and α' are disjoint histories, the probability of the history defined by the union of α and α' ("α or α'") should be the sum of the probabilities of each constituent history:

$$p(\alpha \cup \alpha') = p(\alpha) + p(\alpha') \tag{3.5}$$

For example, suppose that α denotes a set of histories which pass through a series of gates between $X = 0$ and $X = 1$ on the X axis at a series of times, and α' denotes a set of histories passing through gates between $X = 1$ and $X = 2$ at the same times. The histories defined by their union pass through gates between $X = 0$ and $X = 2$ at the same times.

It is easy to see that Eq. (3.5) is not generally satisfied, since

$$\begin{aligned} p(\alpha \cup \alpha') &= \left(\int_\alpha + \int_{\alpha'}\right) \mathcal{D}X(t) \left(\int_\alpha + \int_{\alpha'}\right) \mathcal{D}Y(t) \\ &\quad \times \exp\left(\frac{i}{\hbar} S[X] - \frac{i}{\hbar} S[Y]\right) \rho_0(X_0, Y_0) \\ &= p(\alpha) + p(\alpha') + 2 \,\mathrm{Re}\, D(\alpha, \alpha') \end{aligned} \tag{3.6}$$

where $D(\alpha, \alpha')$ is the decoherence functional,

$$D(\alpha, \alpha') = \int_\alpha \mathcal{D}X(t) \int_{\alpha'} \mathcal{D}Y(t) \exp\left(\frac{i}{\hbar} S[X] - \frac{i}{\hbar} S[Y]\right) \rho_0(X_0, Y_0) \tag{3.7}$$

Loosely speaking, the decoherence functional measures interference between pairs of trajectories, and the presence of the term Re $D(\alpha, \alpha')$ prevents the sum rules from being satisfied. If this term vanishes, however, for $\alpha \neq \alpha'$, then probabilities can be assigned using the formula (3.4). Experience shows that when a mechanism is introduced to cause Re $D(\alpha, \alpha')$ to become diagonal, typically both the real and imaginary parts vanish,

$$D(\alpha, \alpha') = 0 \qquad \text{for} \quad \alpha \neq \alpha' \tag{3.8}$$

a condition referred to as decoherence. In particular, as indicated above, coupling the system to a thermal environment and tracing it out causes the decoherence condition to be approximately satisfied.

The construction of the decoherence functional for a particle linearly coupled to a thermal environment with temperature T and dissipation coefficient γ (the quantum Brownian motion model) has been described in detail elsewhere (see, for example, refs. 35–37 and 15). Here, only the final result is quoted, which is very simple. After tracing out the environmental coordinates, one finds that the decoherence functional takes the form

$$D(\alpha, \alpha') = \int_\alpha \mathcal{D}X \int_{\alpha'} \mathcal{D}Y \, \rho_0(X_0, Y_0) \\ \times \exp\left(\frac{i}{\hbar} \int dt \left(\frac{1}{2} M\dot{X}^2 - \frac{1}{2} M\dot{Y}^2\right) - D \int dt (X - Y)^2\right) \tag{3.9}$$

where $D = 2M\gamma kT/\hbar^2$. For simplicity we consider the case $V(X) = 0$ and the case of negligible dissipation. For macroscopic values of the parameters M, γ, and T, the factor D is exceedingly large, which means that contributions to the path integral from paths with widely differing values of X and Y are strongly suppressed. Hence the decoherence functional will tend to be very small for $\alpha \neq \alpha'$, so there is approximate decoherence. We may therefore assign probabilities to the histories equal to the diagonal elements of the decoherence functional.

Introducing $Q = \frac{1}{2}(X + Y)$, and $\xi = X - Y$, we can carry out the ξ integral, with the result

$$p(\alpha) = \int_\alpha \mathcal{D}Q \, W_0(M\dot{Q}_0, Q_0) \exp\left(-\frac{1}{4\hbar^2 D} \int dt (M\ddot{Q})^2\right) \tag{3.10}$$

where W_0 is the Wigner function of the initial density operator [38]. The interpretation of this result is reasonably clear. The probability distribution is strongly peaked about trajectories in configuration space satisfying the classical equation of motion $\ddot{Q} = 0$. The factor $\hbar^2 D = 2M\gamma kT$ represents thermal fluctuations about deterministic motion, but if the mass of the particle is sufficiently large, these are comparatively small [39]. The Wigner function essentially provides a measure on the initial conditions of the trajectories [40]. Hence, a sufficiently massive particle will behave approximately classically in the presence of a decohering environment of sufficiently large temperature.

Having established the conditions required for the classicality of the large particle, we now couple in the small quantum system. The decoherence functional for the composite three-component system (massive particle, quantum particle, environment), with the environment traced out, is

$$D(\alpha, \alpha') = \int_{\alpha} \mathcal{D}X \int_{\alpha'} \mathcal{D}Y \int \mathcal{D}x\, \mathcal{D}y\, \rho_0^A(X_0, Y_0)\rho_0^B(x_0, y_0)$$
$$\times \exp\left(\frac{i}{\hbar}\int dt\left(\frac{1}{2}M\dot{X}^2 - \frac{1}{2}M\dot{Y}^2\right) - D\int dt(X - Y)^2\right)$$
$$\times \exp\left(\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2 - \lambda Xx\right)\right)$$
$$\times \exp\left(-\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega^2y^2 - \lambda Yy\right)\right) \qquad (3.11)$$

This formula is an elementary generalization of Eq. (3.9). The initial density matrices of the massive and light particle are denoted $\rho^A(X_0, Y_0)$ and $\rho^B(x_0, y_0)$, repectively. The inclusion of the light particle little affects decoherence, so will we assume it, and take the probabilities for the histories of the massive particle to be given by the diagonal elements of Eq. (3.11).

Again introducing $Q = \frac{1}{2}(X + Y)$ and $\xi = X - Y$, we can perform the integration over ξ, with the result, for the probabilities for histories,

$$p(\alpha) = \int_{\alpha} \mathcal{D}Q \int \mathcal{D}x\, \mathcal{D}y\, W_0^A(M\dot{Q}_0, Q_0)\rho_0^B(x_0, y_0)$$
$$\times \exp\left(-\frac{1}{8M\gamma kT}\int dt\left(M\ddot{Q} + \frac{1}{2}\lambda(x + y)\right)^2\right)$$
$$\times \exp\left(\frac{i}{\hbar}\int dT\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2 - \lambda Qx\right)\right)$$
$$\times \exp\left(-\frac{i}{\hbar}\int dT\left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega^2y^2 - \lambda Qy\right)\right) \qquad (3.12)$$

where W_0^A is the Wigner transform of the initial density matrix ρ_0^A. This is the desired answer, but the trick is now to write it in a useful form. In particular, it may be written

$$p(\alpha) = \int_\alpha \mathcal{D}Q \int \mathcal{D}\bar{q}\; W_0^A(M\dot{Q}_0, Q_0) w_Q[\bar{q}(t)] \times \exp\left(-\frac{1}{8M\gamma kT(1-\eta)} \int dt (M\ddot{Q} + \lambda\bar{q})^2\right) \tag{3.13}$$

where

$$w_Q[\bar{q}(t)] = \int \mathcal{D}x\, \mathcal{D}y\; \rho_0^B(x_0, y_0) \exp\left(-\frac{\lambda^2}{8M\gamma kT\eta} \int dt \left(\frac{(x+y)}{2} - \bar{q}\right)^2\right) \times \exp\left(\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 - \lambda Q x\right)\right) \times \exp\left(-\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega^2 y^2 - \lambda Q y\right)\right) \tag{3.14}$$

To achieve the decomposition (3.13), (3.14) we have effectively deconvolved the Gaussian in Eq. (3.13), using the functional integral generalization of the formula

$$\exp(-(x-y)^2) = \int dz \exp\left(-\frac{(x-z)^2}{1-\eta} - \frac{(y-z)^2}{\eta}\right) \tag{3.15}$$

This deconvolution is of course not unique, and η is an arbitrary constant parametrizing this nonuniqueness [although clearly the total probability distribution (3.13) is independent of η].

Written in the form (3.13), the probability distribution has a natural interpretation. Suppose, for simplicitly, that the Wigner function of the large particle is strongly peaked about particular values of Q_0 and $M\dot{Q}_0$. Hence in the absence of the coupling to the small particle, Eq. (3.13) describes a probability distribution for the large particle strongly peaked about a single classical solution with prescribed initial conditions, as outlined above. With the small particle coupled in, however, there is the integration over $\bar{q}(t)$ together with the weight function (3.14). Therefore Eq. (3.13) is the sought-after result: it describes an ensemble of trajectories for the large particle evolving according to the stochastic differential equation

$$M\ddot{Q} + \lambda\bar{q} = 0 \tag{3.16}$$

with a weight function for $\bar{q}$ depending on the initial conditions and dynamics of the small particle. The weight function (3.14) is discussed in some detail in ref. 13. Here, we just make a few comments.

First, it can be shown that $w_Q[\bar{q}]$ is almost the formula (2.5) for continuous quantum measurement of the small particle's trajectory. We therefore have close agreement with the first approach to classical–quantum couplings described in Section 2. It is possible, however, that exact agreement with the continuous quantum measurement theory formula could be obtained by exploring different types of coupling between the large particle, small particle, and environment. Still, Eq. (3.14) is sufficiently close to the continuous quantum measurement formula for us to be able to read off the width of the effective continuous measurement—it is of order $M\gamma kT/\lambda^2$, in agreement with the heuristic argument of Section 2.

Second, it can be shown that $w_Q[\bar{q}]$ is exactly a smeared Wigner functional. The Wigner functional, introduced by Gell-Mann and Hartle [15], is a distribution function on histories which bears the same relation to the decoherence functional that the Wigner function bears to the density operator. Like the Wigner function, the Wigner functional is not always positive. Here, however, we obtained a *smeared* Wigner functional, which, like appropriately smeared Wigner functions, is positive [40, 41].

Finally, the crucial property of $w_Q[\bar{q}]$ is that it kills interferences in the initial state of the quantum particle. Interferences between localized states appear as rapid oscillations in the Wigner functional, but the smearing averages these oscillations to zero. (An analogous phenomenon occurs with the usual Wigner function.) Hence, a superposition of localized states may be effectively replaced by the corresponding mixed state, and the weight function $w_Q[\bar{q}]$ for an initial superposition state may therefore be replaced by a sum of weight functions, one for each localized state. We therefore obtain the same result as the approach of Section 2: the classical particle [which responds to the quantum particle via Eq. (3.16)] sees only one element of a superposition, with some probability.

4. CONCLUSIONS

We have presented two schemes for coupling classical and quantum variables which accommodate nontrivial states of the quantum variables in an intuitively sensible way.

The first scheme, in Section 2, is based on the premise that the interaction between the classical and quantum variables may be regarded as a quantum measurement. The mathematics of continuous quantum measurement theory then fixes the overall structure of the scheme, but an additional physical argument is required to fix the parameter σ describing the precision of the measurement.

The second scheme, in Section 3, involves a more fundamental derivation of the form of the effective equations of motion for a simple system consisting of a large particle coupled to a small particle, and coupled also to a thermal environment in order to produce the decoherence necessary for classicality of the large particle. Both of the schemes lead to the desired form (1.7) of the effective equations of motion of the classical particle, but produce slightly different formulas for the probability of the stochastic term $\bar{x}(t)$. This small difference might be reconciled by a more detailed study of the couplings between the systems present.

The first scheme is more phenomenological, and hence more general. And, as pointed out at the Peyresq-3 meeting by David Finkelstein, it makes clear what the *minimal* requirements are for a model of classical–quantum couplings which improves on the naive mean-field approach. The second scheme is more model dependent, but produces a precise value for the width of the effective "measurement" of the quantum particle by the classical particle, verifying the heuristic analysis of Section 2.

Similar results are obtained with different types of couplings, for example, to momentum or to energy [13]. Obviously an important challenge is to extend to quantum field theories and hence to obtain a generalization of Eq. (1.1). This would mean confronting the difficult issues of covariance and nonrenormalizability.

An essential ingredient in these approaches is the explicit appeal to decoherence in order to ensure the quasiclassical behavior of one of the susbystems. Weaker notions of classicality are sometimes used in this context. For example, it is sometimes argued that a massive particle starting out in a coherent state and evolved unitarily will behave "classically." Aside from the fact that a special initial state is required, the "classical" system is really still quantum, and its quantum nature may be seen if it interacts with another subsystem in a nontrivial superposition state, for then the entire composite system would go into a "nonclassical" superposition. The notion of classicality used here, which follows the decoherent histories literature [14, 15], is more comprehensive, and is the appropriate one for the real physical systems that we observe to be effectively classical.

Although we made heavy use of the decoherent histories approach in characterizing emergent classicality in Section 3, it seems very likely that similar results might be found from other approaches, such as the density matrix approach [42–44] or quantum state diffusion picture [23, 22]. A system similar to that considered in this paper has been analyzed by Zoupas [11] using the quantum state diffusion picture, and a simple spin system by Yu and Zoupas [45]. Furthermore, the theory of continuous quantum measurements used in Section 2 is closely related to the so-called hybrid representation of composite quantum systems [10, 46], and this provides yet another possible framework for examining the emergence of a theory of coupled classical–quantum variables.

Calzetta and Hu [47], in the context of system–environment models (such as quantum Brownian motion), have written down stochastic equations describing the stochastic effect of a thermal environment on the system. They have also discussed the decoherence of "correlation histories" in field theories, and have shown that histories specified by values of the energy-momentum tensor are approximately decoherent, and thus may be assigned probabilities [48]. This leads to the possibility that the right-hand side of (1.1) may be taken to be a stochastic c-number, $\bar{T}_{\mu\nu}$, whose probabilities are given by the expression derived by Calzetta and Hu, thereby generalizing the results discussed here to the full Einstein equations. Some other related work may be found in refs. 49 and 50.

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